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ON THE SUPPORT OF HARMONIC MEASURE FOR THE RANDOM WALK

BY

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ABSTRACT

We show that harmonic measure for the simple random walk on the $n \times \cdots \times n$ cube in the d-dimensional lattice is supported on $o(n^d)$ vertices.

1. Introduction

Let $Q^d(n)$ denote the d-dimensional $n \times n \times \cdots \times n$ cube in the lattice Z^d, i.e.

$$Q^d(n) = \{(x_1, \dots, x_d) : x_i \in 0, \dots, n-1\}.$$

We define $\{S_k\}_{k \geq 0}$, $S_0 = v$ as the Markov chain on $Q^d(n)$ which starts at $v \in Q^d(n)$ and

$$P\{S_{k+1} = w | S_0, \dots, S_k\} = (\text{degree of } S_k)^{-1}$$

if w is adjacent to S_k (i.e. $w - S_k = e_i$ or $-e_i$, where e_i is the i'th coordinate vector) and $= 0$ otherwise. $\{S_k\}_{k \geq 0}$ is called a Simple Random Walk (SRW) on $Q^d(n)$ starting at v. Given a set of vertices A, let $\mu_v(S)$ denote the harmonic measure supported on A for the SRW starting at v. That is, for $S \subset A$, $\mu_v(S)$ is the probability that the first visit of the SRW to A is in S. In this note we prove the following.

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THEOREM: *For any $\epsilon > 0$ there is $N(\epsilon)$ so that for any $n > N(\epsilon)$ and any v and A in $Q^d(n)$, there is a set $S(v, A)$ with no more than ϵn^d vertices such that*

$$\mu_v(S(v, A)) > 1 - \epsilon.$$

Remarks: (1) This result is a discrete high-dimensional analogue of Oksendal's Theorem (Oksendal [8]), asserting that the harmonic measure on compact sets in R^2 is singular to two-dimensional Lebesgue measure. As was pointed out by Carleson, if A is a compact set in R^2 then the density of the harmonic measure at a Lebesgue density point of A is zero, and Oksendal's Theorem follows (see Carleson [3]). Often in random walk, Brownian motion problems, the Brownian motion situation is just a limit of the discrete case. And thus the continuous proof translates to a discrete proof. Oddly, it does not seem that the idea of the proof using Lebesgue density points translates directly to the discrete case, as a discrete variant of the Lebesgue density theorem fails for subsets of the lattice. In the continuous set-up there are by now much better results bounding the Hausdorff dimension of the support of harmonic measure (see Bourgain [2], Bishop [1]).

(2) For a study of harmonic measure on connected sets in Z^2 see Lawler [5]. For background on random walks in Z^d see Lawler [4].

2. Proof

Proof of Theorem: We need the following definition. A vertex u in A is called a **δ-density vertex** if $\forall i = 1, \dots, n$, $|A \cap B(u, i)| > \delta(2i)^d$ where $B(u, r)$ denotes a ball of radius r centered at u in the L^∞ metric. Denote by $\partial B(u, r)$ the set of vertices with distance r from u.

LEMMA 1: *For any A there are at most $2^{2d}\delta n^d$ vertices in A which are not δ-density vertices.*

Proof of Lemma 1: A binary subcube of $Q^d(n)$ is a translation of some cube of the form $Q^d(2^k)$ by a vector $(a_1, \dots, a_d)$, where $\forall i$ a_i is of the form $j2^k$. Note that two binary subcubes are either disjoint or one is contained in the other. Now if a vertex v is not a δ-density vertex then there is a ball $B(v, i)$ centered at v with no more than $\delta(2i)^d$ elements of A. Also there is a binary cube containing v inside $B(u, i)$ with more than $2^{-2d}(2i)^d$ vertices, and no more than $\delta(2i)^d$ elements of

A. So for any vertex which is not a δ-density vertex pick such a binary cube. The union of these binary cubes has a disjoint subcovering. Hence the number of elements of A in the union is smaller than $2^{2d}\delta n^d$, and the lemma follows.

Take $\delta = \epsilon/(2^{2d+1})$; therefore it is enough to estimate the size of the support of harmonic measure restricted to δ-density vertices of A. Denote the set of these vertices by A_δ.

LEMMA 2: *If $u \in A_\delta$ then*

$$\mathbf{P}(\textit{SRW starting anywhere in } B(u,r) \textit{ hits } A \textit{ before hitting } \partial B(u,2r)) > c(\delta)$$

for any $r > 0$.

Proof of Lemma 2: Adapted from Lawler [6] (Lemma 11). Denote by $G'(x,y)$ the Green function for SRW killed upon hitting $\partial B(u,2r)$, i.e. $G'(x,y)$ is the expected number of visits to y for the SRW starting at x and killed on $\partial B(u,2r)$. Then for $d \geq 3$, $\forall x,y \in B(u,r)$, $G'(x,y)/G(x,y) > C$, where $G(x,y)$ is the Green function for SRW in Z^d, and C is independent of r (see Lawler [4, p. 35]). In two dimensions $\forall x,y \in B(u,r)$, $G'(x,y)$ is uniformly bounded away from zero, for all r. Now let V_x denote the number of visits of SRW starting at x to $A \cap B(u,r)$ before hitting $\partial B(u,2r)$. Then

$$E(V_x) = \sum_{y \in (A \cap B(u,r))} G'(x,y) \geq \delta r^d \inf_{y \in (A \cap B(u,r))} G'(x,y).$$

Yet for x,y in $B(u,r)$, $G'(x,y) \geq CG(x,y) \geq c' r^{2-d}$. Hence $E(V_x) \geq c'\delta r^2$. But $E(V_x | V_x \geq 1)$ is smaller than the expected time till hitting $\partial B(u,r)$, and it is standard that the expected time is smaller than $c'' r^2$. Hence

$$\mathbf{P}(V_x \geq 1) = E(V_x)[E(V_x | V_x \geq 1)]^{-1} \geq c'\delta r^2 (c'' r^2)^{-1} = c(\delta) > 0.$$

We are done with the lemma; back to the proof of the theorem. We follow an idea from Bourgain [2].

For simplicity we will assume first that $n = 3^k$ for some k, and that the SRW starts at the boundary of the cube. The adaptation to the general case is easy and will be clear from the proof below. Divide $Q^d(n)$ into 3^{ld} subcubes $\{Q_j\}$ each of size 3^{k-l}. If l is large enough, depending only on the δ in the density condition, then there is a subcube Q_0 such that $\mu_v(Q_0 \cap A_\delta) \leq 1/2(3^{ld})$, i.e., Q_0 gets less than its "fair share" of harmonic measure. To prove this claim we consider two cases. First, if there is a subcube Q_0 such that $Q_0 \cap A_\delta = \emptyset$ then the claim is trivial. Otherwise every subcube contains points of A_δ and so by the density condition SRW starting in such a cube has a chance $c(\delta)$ of being captured by A before it leaves the double of the cube. Let Q_0 be a subcube at the center of $Q^d(n)$ and note that SRW starting on the boundary of the cube must pass through at least $3^l/4$ nonadjacent cubes to get to Q_0, so has less than $(1-c(\delta))^{3^l/4}$ chance of getting there. Since this is much less than $(2(3^{ld}))^{-1}$ for l large, we have proven the claim. Proceed by dividing each subcube again into 3^{ld} subcubes. By the same argument each subcube contains a further subcube getting less than its "fair share", and so on. The iterative subdivision into subcubes results in a 3^{ld}-tree structure among the subcubes. $Q^d(n)$ is the root of the tree. First generation subcubes are the children of the root, and so on. To finish, assume T is a k tree, i.e., each vertex has k children. Let T_m denote the m'th level of the tree, i.e. all vertices that have m edges between them and the root. Identify T_m with $\{0,\dots,k-1\}^m$. Let ν be a subprobability measure on T. That is, the sum of ν on any level of the tree is $c \le 1$, the measure of a vertex equals the sum of the measures of its children, and all values are nonnegative. Further, assume that for any vertex v in the tree, v has a child $u = \{v, 0\}$ for which $\nu(u) < \nu(v)/(2k)$. Pick a random geodesic in the tree according to uniform measure. That is, start at the root, pick with equal probability one of its sons. Once you have picked a son, pick the next vertex uniformly from his sons and so on. Let $\{v_i\}_{1\le i}$ denote that sequence. The Radon–Nikodym derivative is $\nu(v_i)/k^i = X_i\nu(v_{i-1})/k^{i-1}$. For all i, we have

$$P(X_i < 1/2 \,|X_1,\dots,X_{i-1}) \ge 1/k \quad \text{and} \quad E(X_i \,|X_1,\dots,X_{i-1}) = 1.$$

Denote $\theta_i = E(\sqrt{X_i} \,|X_1,\dots,X_{i-1})$. By Chebyshev's inequality

$$\frac{1}{k} \le P(\sqrt{X_i} < \sqrt{1/2} \,|X_1,\dots,X_{i-1}) \le \frac{1-\theta_i{}^2}{(\theta_i - \sqrt{1/2})^2}$$

which implies that θ_i are bounded away from 1 (indeed, $\theta_i \le 1 - 1/40k$).

It follows that $E(\Pi_{1\le i\le n}\sqrt{X_i}) \to 0$, and therefore $\Pi_{1\le i}X_i \to 0$ in probability. (By the martingale convergence theorem, this convergence also holds almost surely.)

By definition $\nu(v_i)/k^i = \Pi_{j=1}^i X_i$; thus ν tends to be supported on $o(k^m)$ vertices of T_m. Translating back to the cube, we get that most of the harmonic measure restricted to the δ-density vertices is supported on $o((3^{ld})^m)$ of the cubes in the m'th generation of the subdivision. ∎

Remarks: (1) As a corollary to the theorem we get that if $n > N(\epsilon)$, $v \in Q^d(n)$ and $A \subset Q^d(n)$, $|A| > \epsilon n^d$ then $\exists u \in A$ $\mu_v(u) < \epsilon/|A| - \epsilon n^d$.

QUESTION: *Show that for any $c > 0$ $\exists f_c$ a superpolynomial function (i.e. grows faster than any polynomial) such that $\forall A \subset Q^d(n)$ $|A| > cn^d$, $\exists u \in A$ $\mu_v(u) < f_c(n)^{-1}$.*

(2) In the theorem above, we first fixed the dimension and then let n go to infinity. Instead, one can ask if a similar result holds once we have fixed $n = 2$, and let d go to infinity. The example below, due to B. Weiss, shows that the analogue of the theorem fails.

Example: Consider $Q^d(2) = \{0,1\}^d$. Let A_d be a subset of $\{0,1\}^d$ consisting of all the vertices with a number of 1's (Hamming weight) between $d/2 - (1/10)d^{1/2}$ and $d/2$, and 1 as first coordinate. $|A_d| > c'2^d$, for some fixed $c' > 0$. Yet for any $v \in A_d$, $\mu_0(v) > C2^{-d}$ for some universal constant C, where 0 denotes the all 0's vector. Here is a sketch why it is so. Order the cube into d levels according to the Hamming weight. Note that μ_0 is uniform on any level. Now look at the first visit of the simple random walk to level $d/2 - (1/10)d^{1/2}$. With probability close to $1/2$ the random walk will first visit this level in a vertex, with a 0 as first coordinate. Conditioning on that event, the random walk will walk more T steps, where T is a geometric r.v. $P(T = k) = ((d-1)/d)^{k-1}(1 - ((d-1)/d))$, till the first coordinate will flip to 1, $E(T) = d$. After T steps, the level the random walk will visit is dominated from above by $d/2 - (1/10)d^{1/2} + N(0,T) + Td^{-1/2}$, where N is a normal random variable, as $d^{-1/2}$ is an upper bound on the drift up in the levels, above level $d/2 - d^{1/2}$. Thus conditioning on $d < T < 2d$ the level in which the walk first visits A_d is distributed almost uniformly.

This suggests the following. Given a monotone function $f\colon N \to N$ consider the family $\{Q^n(f(n))\}_{n \geq 1}$.

QUESTION: *For which f's is most of the harmonic measure always supported on $o(n^{f(n)})$ vertices in $\{Q^n(f(n))\}$?*

We conjecture that all monotone functions f tending to infinity have this property.

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